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LETTER TO THE EDITOR

Method of convex rigid frames and applications in studies of multipartite quNit pure states

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Abstract

In this letter, we suggest a method of convex rigid frames in the studies of multipartite quNit pure states. We illustrate what the convex rigid frames are, and what is their method. As applications, we use this method to solve some basic problems and give some new results (three theorems): the problem of the partial separability of the multipartite quNit pure states and its geometric explanation; the problem of the classification of multipartite quNit pure states, giving a perfect explanation of the local unitary transformations; thirdly, we discuss the invariants of classes and give a possible physical explanation.

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It is known that in quantum mechanics and quantum information, contrasting the case of bipartite quantum systems with that of studies of multipartite quantum systems, the latter is even more difficult. For instance, for general multipartite quantum systems the problems of the criteria of various separability, of the entanglement measures, of the classification and invariants, etc, all are not yet solved satisfactorily. In the studies of the multipartite quantum pure states, we generally always use the traditional method, i.e., we discuss the state vectors or the density matrices in the Hilbert space, etc. However, sometimes this method is not quite effective; in particular, for some problems the results always are short of an explicit or geometric explanation. This urges us to find some non-traditional ways in quantum mechanics and quantum information. The purpose in this letter is just to discuss some problems in this respect.

In this letter, we first illustrate what a convex rigid frame is and we suggest a new method, called the '*method of convex rigid frames*' (see below), which associates a multipartite quNit pure state with a convex polyhedron and its point in the Hilbert–Schmidt (H–S) space (on the real number field all Hermitian operators acting upon a Hilbert space form a linear space, called the Hilbert–Schmidt space). Sometimes, this method is more effective. As examples of some applications, in this letter we use this method to study three basic problems and give some new results (three theorems): the first is the problem of the so-called partial separability of the multipartite quNit pure states and its perfect geometric explanation; secondly we discuss the problem of the classification of the multipartite quNit pure states, and give a perfect geometric explanation of the local unitary transformations; thirdly, we discuss the invariants of classes and give a possible physical explanation.

Sometimes, we call a vector (operator) in H–S space a 'point'. In this letter, the operators (vectors, points) considered by us are all density matrices. In H–S space, the interior product between two vectors A and B is defined as [1] $\langle A, B\rangle = \mathrm{tr}(A^\dagger B)$; the modulus of a vector A is defined by $\|A\| = \sqrt{\langle A, A\rangle} = \sqrt{\mathrm{tr}(A^\dagger A)}$. The distance $d(A, B)$ between two points A and B is defined by $d(A, B) = \|A - B\|$. In H–S space, if a n-convex polyhedron $\mathbf{C}_n$ has n vertices σ_i $(i = 1, \ldots, n)$, then the convex sum $\sigma = \sum_{i=1}^n \lambda_i\sigma_i \left(0 \leqslant \lambda_i \leqslant 1, \sum_{i=1}^n \lambda_i = 1\right)$ denotes a point in $\mathbf{C}_n$; we label this point σ by $(\lambda_i) \equiv (\lambda_1, \ldots, \lambda_n)$. We denote the set of above n vertices $\{\sigma_i\}$ and the fixed point σ together a symbol $\{(\sigma_i), (\lambda_i)\}$. In this letter, for the study of the M-partite quNit pure states, every related convex polyhedron $\mathbf{C}_n$ and the corresponding point σ can only move as a rigid body as in classical mechanics, so we call it a 'n-convex rigid frame (n-CRF)', simply read as CRF $= \{(\sigma_i), (\lambda_i)\}$.

Definition 1. *Two n-convex rigid frames* CRF $= \{(\sigma_i), (\lambda_i)\}$ *and* CRF$' = \{(\sigma_i'), (\lambda_i')\}$ *are called identical, if* $d(\sigma_i, \sigma_j) = d(\sigma_i', \sigma_j')$ *and* $\lambda_i = \lambda_i'$ *for any* $i, j = 1, \ldots, n$. *In this case, we call the process* CRF $\longrightarrow$ CRF$'$ *a 'motion from* CRF *to* CRF$'$'.

*Obviously, this identical relation is an equivalence relation, therefore all n-*CRFs *can be classified by this identical relation.*

Now, we consider a multipartite quantum system $H = \otimes_{i=1}^M H_i$ *with M parties, all local Hilbert spaces* H_i *having the same dimension N; then the total dimensionality of H is* N^M. *Under the standard natural basis* $\{|i_1\cdots i_M\rangle\}$ $(i_k = 0, 1, \ldots, N-1$ and $k = 1, \ldots, M)$, *a normalized M-partite quNit state vector* $|\Psi\rangle \in H$ *has the form*

$$|\Psi\rangle = \sum_{i_1,\ldots,i_M=0}^{N-1} c_{i_1 i_2\cdots i_M}|i_1\cdots i_M\rangle, \qquad c_{i_1 i_2\cdots i_M} \in \mathbb{C}^1, \qquad \sum_{i_1,\ldots,i_M=0}^{N-1} \left|c_{i_1 i_2\cdots i_M}\right|^2 = 1. \tag{1}$$

In the following, we denote the set of all M-partite quNit pure state density matrices $\rho = |\Psi\rangle\langle\Psi|$ *by the symbol* $\mathbb{P}_{M\times N}$; *then* $\mathbb{P}_{M\times N}$ *is a set of points in the* N^{2M}*-dimensional H–S space. For a given* $\rho = |\Psi\rangle\langle\Psi|$, *in the following way we at once can obtain a set of* CRFs. *In the following,* $\mathbb{Z}_M$ *denotes the integer set* $\mathbb{Z}_M = \{1, \ldots, M\}$, *and* $(r)_P$ *denotes a non-null, proper and naturally ordered subset in* $\mathbb{Z}_M$, $(r)_P \subset \mathbb{Z}_M$, $(r)_P \equiv \{r_1, \ldots, r_P\}$, *where* $1 \leqslant P \leqslant M-1, r_1 < \cdots < r_P$, *and we denote the set* $\left[i_{(r)_P}\right] \equiv \left\{i_{r_1}, \ldots, i_{r_P}\right\} \left(i_{r_1}, \ldots, i_{r_P} = 0, \ldots, N-1\right)$. *Now, for a* $|\Psi\rangle$ *as in equation (1) and any fixed set* $\left[i_{(r)_P}\right]$, *we define a* $(M-P)$*-partite quNit pure state* $\left|\Psi\left[i_{(r)_P}\right]\right\rangle$ *by*

$$\left|\Psi\left[i_{(r)_P}\right]\right\rangle = \sum_{i_{s_1},\ldots,i_{s_{M-P}}=0,\ldots,N-1 \text{ for all } s_1,\ldots,s_{M-P}\notin(r)_P} c_{i_1\cdots i_M}|i_1\cdots i_M\rangle \tag{2}$$

i.e., for $\left|\Psi\left[i_{(r)_P}\right]\right\rangle$, *the indices* $i_{r_1}, \ldots, i_{r_P}$ *are fixed, sum up only for the others,* $i_{s_1}, \ldots, i_{s_{M-P}}$. *Note that* $\left|\Psi\left[i_{(r)_P}\right]\right\rangle$, *generally, is not normalized; we make the normalization*

$$\begin{aligned}
&\left|\varphi\left[i_{(r)_P}\right]\right\rangle = \left(\eta_{[i_{(r)_P}]}(\rho)\right)^{-1}\left|\Psi\left[i_{(r)_P}\right]\right\rangle,\\
&\eta_{[i_{(r)_P}]}(\rho) = \sqrt{\sum_{i_{s_1},\ldots,i_{s_{M-P}}=0,\ldots,N-1 \text{ for all } s_1,\ldots,s_{M-P}\notin(r)_P} \left|c_{i_1\cdots i_N}\right|^2}
\end{aligned} \tag{3}$$

where $\eta_{[i_{(r)_P}]}(\rho)$ *is the normalization factor. We write the pure state density matrix* $\sigma_{[i_{(r)_P}]}(\rho) \equiv \left|\varphi\left[i_{(r)_P}\right]\right\rangle\left\langle\varphi\left[i_{(r)_P}\right]\right|$*; (for all possible* $[i_{(r)_P}]$*) their total is* N^P*.*

Now for every pure state density matrix $\rho = |\Psi\rangle\langle\Psi|$*, from the normalization condition of* $|\Psi\rangle$ *we have*

$$\sum_{\text{for all possible}(r)_P} \eta^2_{[i_{(r)_P}]}(\rho) = 1 \tag{4}$$

then

$$\sigma_{(r)_P}(\rho) = \sum_{\text{for all possible}(r)_P} \lambda_{[i_{(r)_P}]}(\rho)\sigma_{[i_{(r)_P}]}(\rho)$$

is a point in the N^P*-convex polyhedron with vertices* $\left\{\sigma\left[i_{(r)_P}\right]\right\}$*, where* $\lambda_{[i_{(r)_P}]} = \eta^2_{[i_{(r)_P}]}(\rho)$*. Thus, for every pure state density matrix* ρ *we always give a corresponding* N^P-CRF *as*

$$\mathrm{CRF}_{(r)_P}(\rho) = \left\{\left(\sigma_{[i_{(r)_P}]}(\rho)\right), \left(\lambda_{[i_{(r)_P}]}(\rho)\right)\right\} \qquad \left(\text{for all possible } \left[i_{(r)_P}\right]\right). \tag{5}$$

Here, we note an interesting fact that every $\mathrm{CRF}_{(r)_P}(\rho)$*, as a matrix, is just equal to the partial trace* $\mathrm{tr}_{(r)_P}(\rho) \equiv \mathrm{tr}_{r_1\cdots r_P}(\rho)$*. In fact, from the definition of the partial traces and equation (5), this conclusion is obvious; however in the method of convex rigid frames,* $\mathrm{CRF}_{(r)_P}(\rho)$ *is always regarded as a* CRF*. Of course, for the distinct* ρ *and* ρ'*, generally,* $\mathrm{CRF}_{(r)_P}(\rho)$ *and* $\mathrm{CRF}_{(r)_P}(\rho')$ *may be distinct. In the following, for a fixed* $(r)_P$*, we use the symbol* $\mathrm{CRF}_{(r)_P} \equiv \left\{\mathrm{CRF}_{(r)_P}(\rho)\right| : \rho \in \mathbb{P}_{M\times N}\right\}$ *which is a set of* CRFs *corresponding to various pure state density matrices* ρ*.*

Theorem 1. *For each fixed proper subset* $(r)_P \equiv \{r_1, \ldots, r_P\} \subset \mathbb{Z}_M(r_1 < \cdots < r_P, 1 \leqslant P \leqslant M-1)$ *there is a 1–1 correspondence* $T_{(r)_P}$ *between the set* $\mathbb{P}_{M\times N}$ *and the set* $\mathrm{CRF}_{(r)_P}$*, symbolize this by* $T_{(r)_P} : \mathbb{P}_{M\times N} \rightleftarrows \mathrm{CRF}_{(r)_P}$*.*

Proof. As mentioned above, by using equation (5) for every pure state ρ there is always a corresponding $\mathrm{CRF}_{(r)_P}(\rho)$; now we define the mapping $T_{(r)_P}$ by

$$T_{(r)_P} : \mathbb{P}_{M\times N} \longrightarrow \mathrm{CRF}_{(r)_P}, \qquad T_{(r)_P}(\rho) = \mathrm{CRF}_{(r)_P}(\rho) \qquad \text{for} \quad \rho \in \mathbb{P}_{M\times N}. \tag{6}$$

If $\rho' \neq \rho$, then $|\Psi\rangle \neq \pm|\Psi\rangle$, this means that there is at least one of N^P real numbers $\lambda_{[i_{(r)_P}]}(\rho)$, or of N^P matrices $\sigma_{[i_{(r)_P}]}(\rho)$ which is different from one of $\lambda_{[i_{(r)_P}]}(\rho')$, or of matrices $\sigma_{[i_{(r)_P}]}(\rho')$, thus $\mathrm{CRF}_{(r)_P}(\rho) \neq \mathrm{CRF}_{(r)_P}(\rho')$.

Conversely, for any $\mathrm{CRF}_{N^P} = \{(\mu_k), (\omega_k)\}$ $(k = 1, \ldots, N^P) \in \mathrm{CRF}_{(r)_P}$, we can take the set $\{i_{r_1}, \ldots, i_{r_P}\}$ $(i_{r_1}, \ldots, i_{r_P} = 0, \ldots, N-1)$ to substitute the set $\{i_1, \ldots, i_P\}$ of indices in the natural order of $\{k\}$, and we can rewrite CRF_{N^P} as $\mathrm{CRF}_{N^P} = \left\{\left(\mu_{[i_{(r)_P}]}\right), \left(\omega_{[i_{(r)_P}]}\right)\right\}$. Suppose that the pure state

$$\mu_{[i_{(r)_P}]} = \left|\xi\left[i_{(r)_P}\right]\right\rangle\left\langle\xi\left[i_{(r)_P}\right]\right|, \left|\xi\left[i_{(r)_P}\right]\right\rangle = \sum_{i_{s_1},\ldots,i_{s_{M-P}}=0}^{N-1} d_{i_{s_1}\cdots i_{s_{M-P}}} \left|i_{s_1}\cdots i_{s_{M-P}}\right\rangle,$$

then we write

$$|\Phi\rangle = \sum_{j_1,\ldots,j_M=0}^{N-1} f_{j_1\cdots j_M}|j_1\cdots j_M\rangle$$

where $f_{j_1\cdots j_M}$ is determined by

$$f_{i_1\cdots i_M} = \mu_{[i_{(r)_P}]} d_{i_{s_1}\cdots i_{s_{M-P}}} \qquad \text{when as a set} \quad (i_1\cdots i_M) = \left(i_{r_1}\cdots i_{r_P}\right) \cup \left(i_{s_1}\cdots i_{s_{M-P}}\right). \tag{7}$$

It can be verified directly that $|\Phi\rangle$ is a M-partite quNit normalized pure state, and we just have $T_{(r)_P}(|\Phi\rangle\langle\Phi|) = \mathrm{CRF}_{N^P}$. □

Since in the above discussion, $(r)_P$ and $(s)_{M-P}$ are completely symmetric in status, thus in the similar way, for the subset $(s)_{M-P}$ we have yet a $T_{(s)_{M-P}} : \mathbb{P}_{M\times N} \rightleftarrows \mathrm{CRF}_{(s)_{M-P}}$. From theorem 1, $\mathbb{P}_{M\times N}$ and the set $\{\mathrm{CRF}_{(r)_P}\}$ are 1–1 corresponding, therefore *some studies of the multipartite quNit pure states can be returned into the studies about* $\{\mathrm{CRF}_{(r)_P}\}$. In this letter, we call this method the *method of convex rigid frames*. Sometimes, this is more effective. As examples of applications, in the following we use this method to study some basic problems.

The first is the partial separability problem. Generally, the common so-called separability, in fact, is the 'full-separability'. For general multipartite systems, the problem becomes even more complex. In fact, there yet is other concept of separability weaker than full-separability, i.e., the partial separability, e.g., for a tripartite qubit pure state ρ_{ABC}, there are the A–BC-separability, B–AC-separability, C–AB-separability, etc [2, 3]. For the relationship to Bell-type inequalities and some criteria of partial separability of the multipartite systems, see [4–6].

In the first place, we need to define strictly what is the partial separability of a multipartite quNit pure state. Regarding this, we must consider the order numbered by the use of the particles. If two ordered proper subsets $(r)_P \equiv \{r_1,\dots,r_P\}(1 \leqslant r_1 < \cdots < r_P \leqslant M)$ and $(s)_{M-P} \equiv \{s_1,\dots,s_{M-P}\}(1 \leqslant s_1 < \cdots < s_{M-P} \leqslant M)$ in $\mathbb{Z}_M$ obey

$$(r)_P \cup (s)_{M-P} = \mathbb{Z}_M,\ (r)_P \cap (s)_{M-P} = \emptyset \tag{8}$$

where P is an integer, $1 \leqslant P \leqslant M-1$, then the set $\{(r)_P, (s)_{M-P}\}$ forms a partition of $\mathbb{Z}_M$; in the following for the sake of stress, we denote it by the symbol $(r)_P\|(s)_{M-P}$. Now, for a given partition $(r)_P\|(s)_{M-P}$, we use the natural basis $\{|i_{r_1}\cdots i_{r_P} i_{s_1}\cdots i_{s_{M-P}}\rangle\}$ and write

$$|\Psi_{(r)_P\|(s)_{M-P}}\rangle \equiv \sum_{i_1,\dots,i_M=0}^{N-1} d_{i_1i_2\cdots i_M}|i_{r_1}\cdots i_{r_P} i_{s_1}\cdots i_{s_{M-P}}\rangle, \qquad d_{i_1i_2\cdots i_M} = c_{i_{r_1}\cdots i_{r_P} i_{s_1}\cdots i_{s_{M-P}}}. \tag{9}$$

Obviously, $|\Psi_{(r)_P\|(s)_{M-P}}\rangle$ and $|\Psi\rangle$ in equation (1), in fact, are completely the same in physics; the only difference is the order numbered by the use of the particles. For instance, $\Psi_{A\|BCD} = \Psi_{AB\|CD} = \Psi_{ABC\|D} = \Psi_{ABCD}$ and $\Psi_{AC\|BD} = \sum c_{ijkl}|i_A k_C j_B l_C\rangle$, etc. However, we note that, generally, $\rho_{(r)_P\|(s)_{M-P}} \equiv |\Psi_{(r)_P\|(s)_{M-P}}\rangle\langle\Psi_{(r)_P\|(s)_{M-P}}| \neq |\Psi\rangle\langle\Psi| = \rho$ under the standard basis $\{|i_1\cdots i_M\rangle\}$, unless $(r)_P\|(s)_{M-P}$ just maintains the natural order of $\mathbb{Z}_M$ (i.e., $(r)_P = (1,\dots,P)$, $(s)_{M-P} = (P+1,\dots,M)$), so $\rho_{(r)_P\|(s)_{M-P}} = \rho$.

Definition 2. *For the partition $(r)_P\|(s)_{M-P}$, a M-partite quNit pure state $|\Psi\rangle$ is called $(r)_P-(s)_{M-P}$-separable, if the corresponding $|\Psi_{(r)_P\|(s)_{M-P}}\rangle$ can be decomposed as a product of two pure states as*

$$|\Psi_{(r)_P\|(s)_{M-P}}\rangle = |\Psi_{(r)_P}\rangle \otimes |\Psi_{(s)_{M-P}}\rangle \qquad \text{or} \qquad \rho_{(r)_P\|(s)_{M-P}} = \rho_{(r)_P} \otimes \rho_{(s)_{M-P}} \tag{10}$$

where

$$|\Psi_{(r)_P}\rangle \in \otimes_{\alpha=1}^{P} H_{r_\alpha},\ \rho_{(r)_P} = |\Psi_{(r)_P}\rangle\langle\Psi_{(r)_P}\rangle$$

and

$$|\Psi_{(s)_{M-P}}\rangle \in \otimes_{\alpha=1}^{M-P} H_{s_\alpha},\ \rho_{(s)_{M-P}} = |\Psi_{(s)_{M-P}}\rangle\langle\Psi_{(s)_{M-P}}|.$$

If $|\Psi\rangle$ is not $(r)_P-(s)_{M-P}$-separable, then we call it $(r)_P-(s)_{M-P}$-inseparable.

We note that for the distinct partitions, ρ can have distinct partial separability. Of course, if a pure state ρ is partially inseparable with respect to any partition, then it must be entangled. Conversely, if a pure state is always completely partially separable with respect to all possible *partitions* $(r)_P\|(s)_{M-P}$*, then it is separable (disentangled, full-separable). By using the above method of* CRFs*, we can obtain the following theorem, which, in fact, is a geometric explanation of the partial separability of the M-partite quNit pure states.*

Theorem 2. *The sufficient and necessary conditions of the M-partite quNit pure state* $\rho = |\Psi\rangle\langle\Psi|$ *to be* $(r)_P - (s)_{M-P}$*-separable is that* $\mathrm{CRF}_{(r)_P}(\rho)$ *(or* $\mathrm{CRF}_{(s)_{M-P}}(\rho)$*) shrinks to one point (pure state vertex), i.e., all*

$$d\big(\sigma_{[i_{(r)_P}]} - \sigma_{[i'_{(r)_P}]}\big) = 0 \qquad \text{(or all} \quad d(\sigma_{[i_{(s)_{M-P}}]} - \sigma_{[i'_{(s)_{M-P}}]}) = 0),$$
$$\text{for any} \quad i, i' = 0, \ldots, N-1.$$

Proof *Necessity***.** Suppose that the pure state $\rho = |\Psi\rangle\langle\Psi|$ is $(r)_P - (s)_{M-P}$-separable, according to definition 2, this means that (see equation (9))

$$\big|\Psi_{(r)_P\|(s)_{M-P}}\big\rangle = \big|\Psi_{(r)_P}\big\rangle \otimes \big|\Psi_{(s)_{M-P}}\big\rangle.$$

If the normalized $\big|\Psi_{(r)_P}\big\rangle$ and $\big|\Psi_{(s)_{M-P}}\big\rangle$, respectively, are

$$\big|\Psi_{(r)_P}\big\rangle = \sum_{i_{r_1},\ldots,i_{r_P}=0}^{N-1} d_{i_{r_1}\cdots i_{r_P}} \big|i_{r_1}\cdots i_{r_P}\big\rangle \quad \text{and} \quad \big|\Psi_{(s)_{M-P}}\big\rangle = \sum_{i_{s_1},\ldots,i_{s_{M-P}}=0}^{N-1} e_{i_{s_1}\cdots i_{s_{M-P}}} \big|i_{s_1}\cdots i_{s_{M-P}}\big\rangle,$$

then by a direct calculation, in $\mathrm{CRF}_{(r)_P}(\rho)$ we have

$$\lambda_{[i_{(r)_P}]}(\rho) = \big|d_{i_{r_1}\cdots i_{r_P}}\big|^2 \qquad \text{and all} \qquad \sigma_{[i_{(r)_P}]}(\rho) = \big|\Psi_{(r)_P}\big\rangle\big\langle\Psi_{(r)_P}\big| \qquad (11)$$

which means that $\mathrm{CRF}_{(r)_P}(\rho)$ will indeed shrink to a point; similarly for $\mathrm{CRF}_{(s)_{M-P}}(\rho)$.

Sufficiency. If all $\sigma_{[i_{(r)_P}]}(\rho)$ shrink to a point

$$\sigma = |\varphi\rangle\langle\varphi|, |\varphi\rangle = \sum_{k_1,\ldots,k_P=0}^{N-1} f_{k_1\cdots k_P}|k_1\cdots k_P\rangle,$$

then according to equations (9) and (10), this means that

$$\big|\Psi_{(r)_P\|(s)_{M-P}}\big\rangle = \sum_{i_{r_1},\ldots,i_{r_P}}^{N-1} f_{k_1\cdots k_P}\big|i_{r_1}\cdots i_{r_P}\big\rangle \otimes \sum_{i_{s_1},\ldots,i_{s_{M-P}}} g_{i_{s_1}\cdots i_{s_{M-P}}}\big|i_{s_1}\cdots i_{s_{M-P}}\big\rangle = |\varphi\rangle \otimes |\psi\rangle,$$

where

$$|\psi\rangle = \sum_{i_{s_1},\ldots,i_{s_{M-P}}} g_{i_{s_1}\cdots i_{s_{M-P}}}\big|i_{s_1}\cdots i_{s_{M-P}}\big\rangle, g_{i_{r_1}\cdots i_{r_P}}$$

are some coefficients, and $|k_1\cdots k_P\rangle$ has been substituted by $\big|i_{r_1}\cdots i_{r_P}\big\rangle$. Therefore ρ is $(r)_P - (s)_{M-P}$-separable. □

Corollary. *For a M-partite quNit pure state* ρ, $\mathrm{CRF}_{(r)_P}(\rho)$ *and* $\mathrm{CRF}_{(s)_{M-P}}(\rho)$ *both shrink to points, or both do not.*

The proof is evident from the proof of theorem 2.

Therefore in view of the method of CRFs, every separable multipartite quNit (disentangled) pure state is an extremely special state, i.e., of which all CRFs must be shrunk to a point. As a simple example, we consider the normalized tripartite qutrit pure state

$$\rho_{3\times3} = |\Psi_{3\times3}\rangle\langle\Psi_{3\times3}| \in \mathbb{P}_{3\times3}, |\Psi_{3\times3}\rangle = \sum_{i,j,k=0}^{2} c_{ijk}|i_A j_B k_C\rangle \qquad \left(\sum_{i,j,k=0}^{2} |c_{ijk}|^2 = 1\right)$$

and the partition $B\|AC$; by a direct calculation we obtain a 3-CRF

$$\mathrm{CRF}_{(B)}(\rho_{3\times 3}) = \left\{\left(\sigma_{[(j_B)]}(\rho_{3\times 3})\right), \left(\lambda_{[(j_B)]}(\rho_{3\times 3})\right)\right\} \qquad (j = 0, 1, 2) \tag{12}$$

where

$$\lambda_{[(j_B)]}(\rho_{3\times 3}) = \sum_{i,k=0}^{2} \left|c_{ij_Bk}^2\right|, \sigma_{[(j_B)]}(\rho_{3\times 3}) = |\varphi_{3\times 2}\rangle\langle\varphi_{3\times 2}|, |\varphi_{3\times 2}\rangle = \left(\lambda_{[(j_B)]}\right)^{-\frac{1}{2}} \sum_{i,k=0}^{2} c_{ij_Bk}|i_A k_C\rangle.$$

It is easily verified that the condition $d\left(\sigma_{[(j_B)]}(\rho^{(3)}) - \sigma_{[(j'_B)]}(\rho^{(3)})\right) = 0\,(j, j' = 0, 1, 2)$ leads to that for any $i, k = 0, 1, 2$ all rates $c_{i0k}{:}c_{i1k}{:}c_{i2k}$ are equal; this is indeed the sufficient and necessary condition for $|\Psi^{(3)}\rangle$ to be B–AC-separable.

Secondly, we study the problem of classification of the *M*-partite quNit pure states. In view of the method of CRFs, a very natural method of classification is to use the motions of the CRFs.

Definition 3. *We say that two M-partite quNit pure states ρ and ρ' are 'equivalent by motion', symbolized by $\rho \backsim \rho'$, if and only if* $\mathrm{CRF}_{(r)_P}(\rho)$ *and* $\mathrm{CRF}_{(r)_P}(\rho')$ *are identical (see definition 1) with respect to all possible non-null proper subsets $(r)_P$ $(1 \leqslant P \leqslant M-1)$.*

A notable advantage of this definition is that this equivalence relation does not break the partial separability of the M-partite quNit pure states. In fact, we have the following

Corollary. *If two M-partite quNit pure states ρ and ρ' are equivalent by motion, then ρ and ρ' both are $(r)_P - (s)_{M-P}$-separable (or both $(r)_P - (s)_{M-P}$-inseparable), with respect to any $(r)_P\|(s)_{M-P}$, i.e., the partial separability is an invariant of class.*

The proof is obvious.

Note that in the above method of classification, we must still solve the problem of reasonableness in physics, because in quantum information a pure state ρ, generally, represents some information status. It is known that, generally, for the indistinguishability of multipartite quNit states we must use the local operation and classical communications (LOCC) [7–9]. Now we prove that our method is reasonable, i.e., we prove that the above classification by motions, in fact, is just the classification of $\mathbb{P}_{M\times N}$ under the local unitary transformations (LUs). In order to prove this, in fact, we only need to prove the following theorem.

Theorem 3. *Two M-partite quNit pure states ρ and ρ' are equivalent by motion (see definition 2), if and only if there are M unitary matrices $u_i(N) \in U(N)\,(i = 1, \ldots, M)$ that*

$$\rho' = u_1(N) \otimes \cdots \otimes u_M(N)\rho u_M^\dagger(N) \otimes \cdots \otimes u_1^\dagger(N). \tag{13}$$

Proof. In the first place, we note that in H–S space only the unitary transformations of operators can keep the invariances of distances and modulus of the vectors, and a tensor product of some unitary matrices is still a unitary matrix. Now, if equation (13) holds, according to equations (2), (3), (5), the change from $\mathrm{CRF}_{(r)_P}(\rho)$ to $\mathrm{CRF}_{(r)_P}(\rho')$ for each $(r)_P$ is determined by a unitary matrix $u_{s_1}(N^{M-P}) \otimes \cdots \otimes u_{s_{M-P}}(N^{M-P})$, which acts upon every 'part' $\left|\Psi\left[i_{(r)_P}\right]\right\rangle$ of $|\Psi\rangle$ in equation (2) and keeps $\eta_{[i_{(r)_P}]}(\rho)$ to be invariant; thus the identical relation between $\mathrm{CRF}_{(r)_P}(\rho)$ and $\mathrm{CRF}_{(r)_P}(\rho')$ is quite obvious.

Conversely, since $\lambda_{[i_{(r)_P}]}(\rho) = \lambda_{[i_{(r)_P}]}(\rho')$ for all possible $\left[i_{(r)_P}\right]$, we know that for every $[i_{(r)_P}]$ there must be a unitary matrix $u_{[i_{(r)_P}]}(N^{M-P})$ which acts upon the 'part' $\left|\Psi\left[i_{(r)_P}\right]\right\rangle$ of $|\Psi\rangle$, and keeps that all equations $d\left(\sigma_{[i_{(r)_P}]}(\rho), \sigma_{[i'_{(r)_P}]}(\rho)\right) = d\left(\sigma_{[i_{(r)_P}]}(\rho'), \sigma_{[i'_{(r)_P}]}(\rho')\right)$ always hold. This fact must hold for arbitrary $(r)_P$ and arbitrary set $[i_{(r)_P}]$ of indices; the unique possibility is that there are some $u_1(N), \ldots, u_M(N), u_k(N) \in U(N)\,(k = 1, \ldots, M)$ and $|\Psi'\rangle = u_1(N) \otimes \cdots \otimes u_M(N)|\Psi\rangle$. □

This theorem gives us a perfect explanation of the LUs (a LU acting upon ρ, in fact, is a motion of CRFs) as a motion of a rigid body as in the classical mechanics.

Thirdly, we discuss the invariants of the classification and a possible explanation. Since the distance between two points (vectors) in H–S space is invariant under any motion (LU), evidently there are at least two kinds of invariants of motions (LUs): the volumes of the convex polyhedrons propped up by the CRFs; others are the angles of intersections of any two 'props' in every CRF.

As for the problem of how to calculate the volume of a convex polyhedron in H–S space, see [10] and its references. For a M-partite quNit pure state ρ and a given $(r)_P\|(s)_{M-P}$, let $V(\mathrm{CRF}_{(r)_P}(\rho))$ denote the volumes of the convex polyhedron with N^P vertices $\{\sigma_{[i_{(r)_P}]}\}$ (for all possible $i_{(r)_P}]$) as in equation (5). Similarly, $V(\mathrm{CRF}_{(s)_{M-P}}(\rho))$. Now we denote the pair of volumes by

$$V_{(r)_P\|(s)_{M-P}}(\rho) = \left[V\left(\mathrm{CRF}_{(r)_P}(\rho)\right), V\left(\mathrm{CRF}_{(s)_{M-P}}(\rho)\right)\right]. \tag{14}$$

Obviously, $V_{(r)_P\|(s)_{M-P}}(\rho)$ is invariant under motions (LUs) of ρ. In addition, in $\mathrm{CRF}_{(r)_P}(\rho)$, the direction from the point $\sigma_{(r)_P}(\rho) = \left\{\lambda_{[i_{(r)_P}]}(\rho)\right\}$ (for all possible $[i_{(r)_P}]$) to a fixed vertex $\sigma\left[k_{(r)_P}\right]$ $(k = 0, \ldots, N-1)$ of $\mathrm{CRF}_{(r)_P}(\rho)$ can be expressed by the vector

$$\omega_{[k_{(r)_P}]}(\rho) = \sum_{\text{for all possible } [i_{(r)_P}]} \left(\lambda_{[i_{(r)_P}]}(\rho) - \delta_{[i_{(r)_P}],[k_{(r)_P}]}\right)\sigma_{[i_{(r)_P}]}(\rho). \tag{15}$$

Therefore, the angle (we label it by $\theta(\rho, [k_{(r)_P}], [l_{(r)_P}])$) of intersection of two directs $\omega_{[k_{(r)_P}]}(\rho)$ and $\omega_{[l_{(r)_P}]}(\rho)$ can be determined by

$$\cos\theta\left(\left[k_{(r)_P}\right], \left[l_{(r)_P}\right], \rho\right) = \left\langle\omega_{[k_{(r)_P}]}(\rho), \omega_{[l_{(r)_P}]}(\rho)\right\rangle \big/ \left\|\omega_{[k_{(r)_P}]}(\rho)\right\| \cdot \left\|\omega_{[l_{(r)_P}]}(\rho)\right\|. \tag{16}$$

Obviously, $\cos\theta\left(\left[k_{(r)_P}\right], \left[l_{(r)_P}\right], \rho\right)$ is still an invariant under motions (LUs) of ρ.

At present, we cannot yet understand the meaning of $\cos\theta\left(\rho, \left[k_{(r)_P}\right], \left[l_{(r)_P}\right]\right)$ in quantum information. However, we find a quite natural explanation of $V_{(r)_P\|(s)_{M-P}}(\rho)$ as follows. From theorem 2, its corollary and the fact that a convex polyhedron shrinks to a point if and only if its volume vanishes, then we know that ρ is $(r)_P - (s)_{M-P}$-separable if and only if $V_{(r)_P\|(s)_{M-P}}(\rho) = (0, 0)$. Conversely, if $V_{(r)_P\|(s)_{M-P}}(\rho) \neq (0, 0)$ then ρ is $(r)_P - (s)_{M-P}$-inseparable, where the value of $V\left(\mathrm{CRF}_{(r)_P}(\rho)\right)$ means that the degree of the difficulty of the factor $\rho_{(r)_P}$ to be separated out from ρ. Similarly, for $V\left(\mathrm{CRF}_{(s)_{M-P}}(\rho)\right)$. Therefore we can regard that $V_{(r)_P\|(s)_{M-P}}(\rho)$ denotes the degree of the measure of the $s_{(r)_P} - s_{(s)_{M-P}}$-inseparability. It is quite interesting that, generally, $V\left(\mathrm{CRF}_{(r)_P}(\rho)\right) \neq V\left(\mathrm{CRF}_{(s)_{M-P}}(\rho)\right)$ unless they both vanish; this means that the above two degrees of difficulty can be different. In addition, it is known that if ρ is $(r)_P - (s)_{M-P}$-inseparability, then there must be the so-called partial entanglement [4, 6]. It is a pity that an entanglement measure, generally, should be in the form of a Neumann entropy and must at least obey some limits [11]; however $V_{(r)_P\|(s)_{M-P}}(\rho)$ has no such properties, so we cannot take it as a measure of partial entanglement.

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